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Constructive Proofs for Approximation by Inner Functions

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1. INTRODUCTION

An inner function is a function on the unit circle T whose values almost everywhere have modulus 1 and are the radial limits of a bounded holomorphic function on the open unit disk U. Recently, Douglas and Rudin [3] proved that, given a function f which is Lebesgue measurable and essentially bounded on T and given $\epsilon > 0$, there exist inner functions $\varphi_1, \psi_1, \varphi_2, \psi_2, \ldots, \varphi_n, \psi_n$ and constants $c_1, \ldots, c_n$ such that

$$\left| f(e^{i\theta}) - \lim_{r\to 1^-} \sum_{k=1}^{n} c_k \frac{\varphi_k(re^{i\theta})}{\psi_k(re^{i\theta})} \right| < \epsilon \tag{1.1}$$

a.e. on T. In the present paper we give a constructive proof of this result. More specifically, we prove that if f is unimodular on T, then we can take, in (1.1), $n = 1$, $c_1 = 1$, while if f is an arbitrary essentially bounded function on T, we can take $n = 2$ and $c_1 = c_2 = \frac{1}{2} \operatorname{ess\,sup}_{(z\in T)} |f(z)|$. We also give another constructive proof for the case that f is continuous on T, and we prove that the inner functions ψ_k in (1.1) can be chosen so that they do not have any zeros in U.

In Section 3 we conclude with some remarks concerning the application of the results obtained to the approximate solution of Wiener–Hopf equations.

2. CONSTRUCTIVE PROOFS

We first establish our notation, which is similar to that in [3].

Let $L^\infty(T)$ denote the set of all bounded complex functions f on the unit circle T for which $f(e^{i\theta})$ is Lebesgue measurable in $0 \leqslant \theta \leqslant 2\pi$. We denote by $\|f\|_\infty$ the essential suprenum of $|f|$ on T, where $f \in L^\infty(T)$. A function

$f \in L^\infty(T)$ is *unimodular* if $|f| = 1$ a.e. on T. The class H^∞ is the set of all $f \in L^\infty(T)$ for which $a_{-n} = 0$ if $n > 0$, where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta, \qquad n = 0, \pm 1, \pm 2, \dots . \tag{2.1}$$

Each function $f \in H^\infty$ is thus given, almost everywhere, by the radial limit of a function that is holomorphic and bounded on the open unit disk U. A function $f \in H^\infty$ is called an *inner function* if f is unimodular.

The following result, essential in our construction, is well known.

LEMMA 2.1. *Let $0 < R_1 < R_2 < \infty$. Let T be the union of disjoint measurable subsets E_1 and E_2 and let u be a function defined on T such that $u = R_j$ on E_j. Then the function*

$$h(z) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log u(e^{i\theta})\, d\theta \right\} \qquad (z \in U) \tag{2.2}$$

is holomorphic in U, satisfies there $R_1 < |h(z)| < R_2$, and the radial limits of h have modulus R_j a.e. *on E_j.*

For example, let $R_1 = R^{-1}$, $R_2 = R$. Let $E_1 = \{e^{i\theta} \mid \theta_1 \leqslant \theta < \theta_2\}$, $E_2 = T - E_1$. In view of Fig. 1, an easy computation yields the following value for $h(z)$:

$$h(z) = R^{1+[(\alpha-2\beta)/\pi]+[(2i/\pi)\ln b/a]}. \tag{2.3}$$

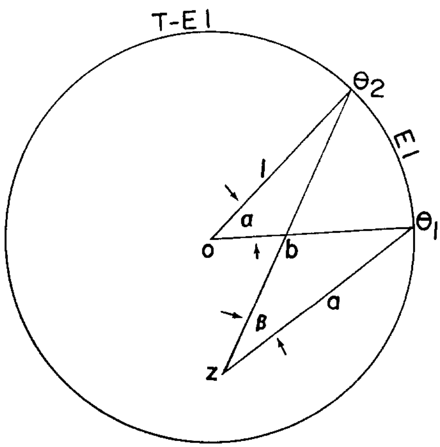

Figure 1. The decomposition of T for Equation (2.3).

Let $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ and let $\eta_1 > 0$, $\eta_2 > 0$ be such that the two circular slits

$$\begin{aligned} S_1 &= \{e^{i\theta} \mid \theta_1 - \eta_2 \leqslant \theta \leqslant \theta_1 + \eta_1\} \\ S_2 &= \{e^{i\theta} \mid \theta_2 - \eta_2 \leqslant \theta \leqslant \theta_2 + \eta_2\} \end{aligned} \tag{2.4}$$

do not overlap. We set

$$\begin{aligned} a &= e^{i(\theta_1+\theta_2)/2}, \\ A &= \tan[(\theta_2 - \theta_1 - \eta_1)/4], \\ k &= \tan[(\theta_2 - \theta_1 - \eta_1)/4]/\tan[(\theta_2 - \theta_1 + \eta_2)/4], \end{aligned} \tag{2.5}$$

Remark. A case of practical importance occurs when $\theta_1 = 0$, $\theta_2 = \pi$. In this case we choose η_1, η_2 and k so that $k^{1/2} = \tan[(\pi - \eta_1)/4] = 1/\tan[(\pi + \eta_2)/4]$ and thus:

$$\begin{aligned} a &= i, \\ A &= k^{1/2}, \\ k &= \tan[(\pi - \eta_1)/4]/\tan[(\pi + \eta_2)/4]. \end{aligned} \tag{2.5'}$$

We use the following standard notation for elliptic functions:

$$\begin{gathered} \mathrm{sn}^{-1}(x; k) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}, \\ K = K(k) = \mathrm{sn}^{-1}(1; k) \\ k' = \sqrt{1 - k^2}, \qquad K' = K(k'), \end{gathered} \tag{2.6}$$

and set

$$\begin{aligned} R &= \exp\{\pi K/K'\}, \\ D(R_1, R_2) &= \{\xi \mid R_1 < |\xi| < R_2\}, \\ D[R_1, R_2] &= \{\xi \mid R_1 \leqslant |\xi| \leqslant R_2\}. \end{aligned} \tag{2.7}$$

We prove

LEMMA 2.2. *Let $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$, and let $\eta_1 > 0$, $\eta_2 > 0$ be such that the circular slits S_j, defined in* (2.4), *do not overlap. The function*

$$z = \Phi(\xi) = \frac{a\{1 + iA\,\mathrm{sn}[K'/\pi \log \xi; k]\}}{1 - iA\,\mathrm{sn}[K'/\pi \log \xi; k]}, \tag{2.8}$$

where a, A and k are defined in (2.5), *maps $D(R_1, R_2) = D(R^{-1}, R)$ conformally onto the z-plane minus the slits $S_j (j = 1, 2)$. As $|\xi|$ approaches R_j, $\Phi(\xi)$ approaches a point of $S_j (j = 1, 2)$. The function $\Phi(\xi)$ is regular in*

$D(R^{-1}, R)$ except for a simple pole at $\xi = e^{-i\beta}$, where β is the smallest positive root of

$$A \operatorname{sn}\left[\frac{iK'}{\pi}\beta; k\right] = 1. \tag{2.9}$$

Remark. In the case of (2.5′) we have $\beta = \pi/2$, i.e., the pole is at $\xi = -i$.

Proof. The transformation

$$\xi = i\frac{a - z}{a + z}; \qquad z = \frac{a(1 + i\zeta)}{1 - i\zeta} \tag{2.10}$$

maps the z-plane minus the circular slits S_j conformally onto the ζ plane minus the slits

$$\begin{aligned} \mathcal{S}_1 &= \{\zeta = \tan[(2\theta - \theta_1 - \theta_2)/4] \mid \theta_1 - \eta_2 \leqslant \theta \leqslant \theta_1 + \eta_1\}, \\ \mathcal{S}_2 &= \{\zeta = \tan[(2\theta - \theta_1 - \theta_2)/4] \mid \theta_2 - \eta_1 \leqslant \theta \leqslant \theta_2 + \eta_2\}, \end{aligned} \tag{2.11}$$

since the circle $z = e^{i\theta}$ is mapped onto the real line $\zeta = \tan[(2\theta - \theta_1 - \theta_2)/4]$.

The map [6, p. 192]

$$\zeta = A \operatorname{sn}\left[\frac{K'}{\pi}\log \zeta; k\right], \tag{2.12}$$

where A is defined by (2.5) (or (2.5′) when $\theta_1 = 0$, $\theta_2 = \pi$), maps $D(R^{-1}, R)$ conformally onto the ζ-plane minus the slits $\mathcal{S}_j$. We now use the second relation (2.10) to obtain (2.8).

Clearly, if we set $\xi = e^{-i\beta}$, where β is defined by (2.9), then $\Phi(\xi) = \infty$. Furthermore, each of the maps used to construct $\Phi(\xi)$ is conformal; hence, $\xi = e^{-i\beta}$ is a simple pole, and there is no other pole.

This completes the proof of Lemma 2.2.

LEMMA 2.3. *Let β be as in Lemma 2.2. Let $k_0 \in (0, 1)$ be the unique solution of the equation*

$$R = \exp[\pi K_0'/(4K_0)], \tag{2.13}$$

where $K_0 = K(k_0)$, $K_0' = K'(k_0)$. Then the function

$$w = \Phi_2(\xi) = \sqrt{k_0}\operatorname{sn}\left[\frac{2iK_0}{\pi}\log(\xi e^{i\beta}); k_0\right] \tag{2.14}$$

is holomorphic in $D[R^{-1}, R]$, maps this closed region onto $|w| \leqslant 1$, has simple zeros at $\xi = \pm e^{-i\beta}$, and satisfies

$$|w(Re^{i\theta})| = |w(R^{-1}e^{i\theta})| = 1, \qquad 0 \leqslant \theta \leqslant 2\pi. \tag{2.15}$$

Proof. The map

$$v = \tfrac{1}{2}(\xi + \xi^{-1}) \tag{2.16}$$

takes $D[R^{-1}, R]$ onto the ellipse[1]

$$\epsilon_R = \left\{ v \mid v = \frac{1}{2}(\rho + \rho^{-1}) \cos t + \frac{i}{2}(\rho - \rho^{-1}) \sin t; \right.$$
$$\left. R^{-1} \leqslant \rho \leqslant R, 0 \leqslant t < 2\pi \right\}. \tag{2.17}$$

The transformation [6, p. 77]

$$w = \sqrt{k_0}\,\mathrm{sn}\left[\frac{2K_0}{\pi} \sin^{-1} v; k_0\right], \tag{2.18}$$

where k_0 is defined by (2.13), maps ϵ_R conformally onto $|w| \leqslant 1$.

If we now note that

$$\tfrac{1}{2}(\xi + \xi^{-1}) = \cosh(\log \xi) = \cos(i \log \xi) = \sin\left(\frac{\pi}{2} - i \log \xi\right) \tag{2.19}$$

and substitute this onto (2.18), we find that the function

$$w_2 = \sqrt{k_0}\,\mathrm{sn}\left[K_0 - \frac{2iK_0}{\pi} \log \xi; k_0\right] \tag{2.20}$$

maps $D[R^{-1}, R]$ onto $|w_2| \leqslant 1$ so that $|w_2(R^{-1}e^{i\theta})| = |w_1(Re^{i\theta})| = 1$. Furthermore, if we replace ξ by ξe^{it}, in (2.20), we obtain a function with the same property. In particular, if we take $t = \beta - \pi/2$, we obtain the function (2.14).

If we set $\xi = \pm e^{-i\beta}$ in (2.14), we obtain $w = 0$. By differentiating (2.14) with respect to ξ we obtain

$$\left|\frac{dw}{d\xi}\right|^2 = \left|\frac{2i\sqrt{k_0}K_0}{\pi\xi} e^{i\beta}\right|^2 |(1 - \mathrm{sn}^2 u)|\,|1 - k_0{}^2 \mathrm{sn}^2 u|, \tag{2.21}$$

where u denotes the quantity in square brackets in (2.14). Setting $\xi = \pm e^{-i\beta}$, we find that

$$\left|\frac{dw}{d\xi}(\pm e^{-i\beta})\right| = 2\sqrt{k_0}K_0/\pi \neq 0,$$

and, hence, w has a simple zero at $\pm e^{-i\beta}$.

[1] The map (2.16) is not a one-to-one map of the closed region $D[R^{-1}, R]$ onto the ellipse ϵ_R. Rather, the ellipse ϵ_R is covered twice: once by the map of $1 \leqslant |\xi| \leqslant R$ and once by the map of $R^{-1} \leqslant |\xi| \leqslant 1$.

THEOREM 2.4. *Let $\epsilon > 0$ be given. Let E_1 and E_2 be disjoint measurable subsets of T whose union is T, and let $\lambda_1 = e^{i\theta_1}$ and $\lambda_2 = e^{i\theta_2}$ be complex numbers of modulus 1, where $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$. Then there exist inner functions φ_1 and φ_2 such that*

$$|\lambda_j - [\varphi_1(e^{i\theta})/\varphi_2(e^{i\theta})]| < \epsilon \quad \textit{for almost all } \theta \textit{ satisfying } e^{i\theta} \in E_j, \quad j = 1, 2.$$

Such functions φ_1, φ_2 are given by

$$\varphi_2(z) = \sqrt{k_0}\,\mathrm{sn}\left[\frac{-2\beta K_0}{\pi} + \frac{iK_0}{\pi^2}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log u(e^{i\theta})\,d\theta; k_0\right], \tag{2.22}$$

$$\varphi_1(z) = \frac{a\left\{1 + iA\,\mathrm{sn}\left[\frac{K'}{2\pi^2}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log u(e^{i\theta})\,d\theta; k\right]\right\}}{1 - iA\,\mathrm{sn}\left[\frac{K'}{2\pi^2}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log u(e^{i\theta})\,d\theta; k\right]}\cdot\varphi_2(z), \tag{2.23}$$

where u, A, k, k_0, a, R_1, R_2 and β are defined as in Lemmas 2.1, 2.2, and 2.3, and where η_1 and η_2, employed in the definition of the slits S_1, S_2, are chosen to be $< \epsilon$.

Proof. Choose the function u appearing in (2.22) and (2.23) as in Lemma 2.1, where $R_1 = 1/R$, $R_2 = R$. Set $\Phi_1(\xi) = \Phi(\xi)\,\Phi_2(\xi)$, where Φ is given in (2.8) and Φ_2 in (2.14). By Lemma 2.1, $h : U \to D(R^{-1}, R)$, and by Lemmas 2.2 and 2.3, both Φ and Φ_2 have modulus 1 on the boundary of $D(R^{-1}, R)$, which implies the same for Φ_1, so that $\Phi_j : D(R^{-1}, R) \to U$. Since h has radial limits R_j a.e. on T, it follows that $\varphi_j(z) = \Phi_j(h(z))$ are inner functions. Setting $\varphi = \varphi_1/\varphi_2$, it follows by our construction that $\lim_{r\to 1^-} \varphi(re^{i\theta}) \in S_j$ for almost every $e^{i\theta} \in E_j$.

This completes the proof of Theorem 2.4.

THEOREM 2.5 (Douglas–Rudin). *The set of all quotients of inner functions is norm-dense in the set of all unimodular functions in $L^\infty(T)$.*

Proof. Let f be a given function in $L^\infty(T)$ which is unimodular, and let $\epsilon > 0$ be given. We divide T into $n(>2)$ equal arcs

$$S^{(j)} = \left\{e^{i\theta} \mid (j - \tfrac{1}{2})\frac{2\pi}{n} < \theta \leqslant (j + \tfrac{1}{2})\frac{2\pi}{n}\right\}, \qquad j = 0, 1, \ldots, n-1, \tag{2.25}$$

and we define

$$E^{(j)} = \{e^{i\theta} \in T \mid f(e^{i\theta}) \in S_j\}; \tag{2.26}$$

n is chosen so that $4\pi/n < \epsilon$. Let $0 < \delta < \epsilon/[2(n-2)\,e]$, $(n-2)\,\delta < 1$, and let

$$\varphi^{(j)}(z) = \frac{\varphi_1^{(j)}(z)}{\varphi_2^{(j)}(z)}, \qquad j = 1, 2, \ldots, n-1, \tag{2.27}$$

denote the ratio of inner functions constructed as in Theorem 2.4 which approaches a point of $S^{(j)}$ within δ of $e^{2\pi ij/n}$ as z approaches any point of $E^{(j)}$, and which approaches a point of $S^{(0)}$ within δ of 1 as z approaches a point of $E^{(0)}$. The function

$$\Phi(z) = \frac{\prod_{j=1}^{n-1} \varphi_1^{(j)}(z)}{\prod_{j=1}^{n-1} \varphi_2^{(j)}(z)} \tag{2.28}$$

is clearly the ratio of two inner functions. For $f(e^{i\theta}) \in S^{(j)}$, the function Φ satisfies

$$\begin{aligned} &| f(e^{i\theta}) - \lim_{r\to 1^-} \Phi(re^{i\theta})| \\ &= \left| f(e^{i\theta}) - \varphi^{(j)}(e^{i\theta}) \prod_{k=1, k\neq j}^{n-1} \varphi^{(k)}(e^{i\theta}) \right| \\ &\leqslant | f(e^{i\theta}) - \varphi^{(j)}(e^{i\theta})| + \left| \varphi^{(j)}(e^{i\theta}) \left[1 - \prod_{k=1, k\neq j}^{n-1} \varphi^{(k)}(e^{i\theta})\right]\right| \\ &\leqslant \frac{2\pi}{n} + (1+\delta)^{n-2} - 1 \qquad \text{a.e.} \quad \text{on } E^{(j)}, \end{aligned} \tag{2.29}$$

since $| f(e^{i\theta}) - \varphi^{(j)}(e^{i\theta})| \leqslant 2\pi/n$ a.e. on $E^{(j)}$, and since $| \varphi^{(k)}(e^{i\theta}) - 1 | \leqslant \delta$ a.e. on $T - E^{(k)}$. Thus the extreme left of (2.29) is bounded almost everywhere by

$$\begin{aligned} \frac{2\pi}{n} + e^{(n-2)\delta} - 1 &\leqslant \frac{2\pi}{n} + (n-2)\,\delta e^{(n-2)\delta} \\ &\leqslant \frac{2\pi}{n} + (n-2)\,\delta e \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{2.30}$$

Notice that our proof establishes

COROLLARY 2.6. *Given any $f \in L^\infty(T)$ which is unimodular on T, and given any $\epsilon > 0$, there exist inner functions φ and ψ such that*

$$\left| f(e^{i\theta}) - \lim_{r\to 1^-} \frac{\varphi(re^{i\theta})}{\psi(re^{i\theta})} \right| < \epsilon \tag{2.31}$$

a.e on T.

THEOREM 2.7 (Douglas–Rudin). *Let Q be the set of all functions of the form $\psi\bar{\varphi}$, where ψ is a finite linear combination of inner functions and φ is inner. Then Q is norm-dense in $L^\infty(T)$.*

Proof. Let $f \in L^\infty(T)$ and set

$$M = \|f\|_\infty, \qquad m = (\|f^{-1}\|_\infty)^{-1}. \tag{2.32}$$

Let $\delta > 0$ be given, and let n and N be integers such that[2]

$$h_1 = (M - m)/n < \delta/2, \qquad h_2 = 2\pi/N < \delta/(2M).$$

For $s = 1, 2,..., n$; $t = 1, 2,..., N$, let

$$R_{s,t} = \{w = re^{i\theta} \mid m + (s-1)\, h_1 \leqslant r < m + sh_1\, ; (t-1)\, h_2 \leqslant \theta < th_2\}, \tag{2.33}$$

$$E_{s,t} = \{e^{i\theta} \in T \mid f(e^{i\theta}) \in R_{st}\}. \tag{2.34}$$

Let S_1, S_2 be defined by (2.4), where $\theta_1 = 0$, $\theta_2 = \pi$, $\eta = \delta(nNM)$, and let

$$G_{s,t}(z) = \frac{\omega_{st}(z)}{\psi_{st}(z)} \tag{2.35}$$

denote the ratio of the inner functions constructed as in Theorem 2.4 which approaches S_j a.e. as z approaches $E_j (j = 1, 2)$, where $E_1 = E_{s,t}$, $E_2 = T - E_{s,t}$. Here we take w, A and k to be defined by (2.5′).

We note that 1 is also the ratio of two inner functions, that $\frac{1}{2}[G_{s,t}(e^{i\theta}) + 1]$ approximates the characteristic function $\chi_{E_{s,t}}$ of $E_{s,t}$, and that

$$\| \chi_{E_{st}}(e^{i\theta}) - \tfrac{1}{2}[G_{st}(e^{i\theta}) + 1] \|_\infty \leqslant \frac{\delta}{nNM}. \tag{2.36}$$

Upon taking w_{st} to be the centroid of R_{st}, it follows that the function

$$\begin{aligned} G^{(n,N)}(z) &= \frac{1}{2} \sum_{s,t} w_{st}[G_{st}(z) + 1] \\ &= \frac{1}{2}\, \frac{\sum_{s,t} w_{s,t} \prod_{s' \neq s, t' \neq t} \psi_{s',t'}(z)[\varphi_{s,t}(z) + \psi_{s,t}(z)]}{\prod_{s',t'} \psi_{s',s'}(z)} \end{aligned} \tag{2.37}$$

is a linear combination of ratios of inner functions which satisfies

$$\begin{aligned} &| f(e^{i\theta}) - \lim_{r\to 1^-} G^{(n,N)}(re^{i\theta})| \ (e^{i\theta} \in E_{st}) \\ &\leqslant | f(e^{i\theta}) - w_{st} | + \frac{1}{2} \sum_{s',t'} | w_{st} | \, \eta < \frac{\delta}{2} + \frac{1}{2} nNM\eta < \delta \end{aligned} \tag{2.38}$$

a.e. on T.

[2] Without loss of generality we assume that $M > m$.

We give now an alternative proof of Theorem 2.7, which is essentially due to Rudin,[3] and which shows that in the approximation of f, two ratios of inner functions suffice.

THEOREM 2.8. *Let $f \in L^\infty(T)$, and let $\epsilon > 0$ be given. Then there exist inner functions φ_j and ψ_j, $j = 1, 2$, such that*

$$\left| f(e^{i\theta}) - \lim_{r\to 1^-} \frac{\|f\|_\infty}{2} \left\{ \frac{\varphi_1(re^{i\theta})}{\psi_1(re^{i\theta})} + \frac{\varphi_2(re^{i\theta})}{\psi_2(re^{i\theta})} \right\} \right| < \epsilon \tag{2.39}$$

a.e. on T.

Proof. If $\|f\|_\infty = 0$, the result is trivial, since we may then choose φ_j and ψ_j arbitrarily. For the remainder of the proof, we shall assume that $\|f\|_\infty > 0$.

Let $z = re^{i\theta}$, $0 \leqslant r \leqslant 1$, and let

$$u = \theta - \arccos r, \qquad v = \theta + \arccos r, \tag{2.40}$$

where we assume that $0 \leqslant \arccos r \leqslant \pi/2$, and where we set $u = 0$, $v = \pi$ if $r = 0$. Then the range of the functions

$$\alpha(z) = e^{iu}, \qquad \beta(z) = e^{iv} \tag{2.41}$$

is T, and the functions

$$\alpha(f/\|f\|_\infty), \qquad \beta(f/\|f\|_\infty) \tag{2.42}$$

are unimodular functions in $L^\infty(T)$. By Corollary 2.6, there are inner functions φ_1, ψ_1, φ_2, and ψ_2 such that

$$\begin{aligned} \left| \alpha(f(e^{i\theta})/\|f\|_\infty) - \lim_{r\to 1^-} \frac{\varphi_1(re^{i\theta})}{\psi_1(re^{i\theta})} \right| &< \frac{\epsilon}{\|f\|_\infty}, \\ \left| \beta(f(e^{i\theta})/\|f\|_\infty) - \lim_{r\to 1^-} \frac{\varphi_2(re^{i\theta})}{\psi_2(re^{i\theta})} \right| &< \frac{\epsilon}{\|f\|_\infty}, \end{aligned} \tag{2.43}$$

a.e. on T. Since

$$f = \frac{\|f\|}{2} \{\alpha(f/\|f\|_\infty) + \beta(f/\|f\|_\infty)\}, \tag{2.44}$$

we obtain (2.39) from (2.43) and (2.44).

It is not always easy in practice to find the sets $E_{s,t}$ corresponding to the R_{st}, nor is it easy to evaluate the integral in (2.2). We therefore develop a more explicit construction in terms of Riemann integrals. The function φ_2

[3] Private communication.

of Theorem 2.4 has several disadvantages from the point of view of an explicit construction, since we cannot explicitly express k_0 of (2.13). Therefore, we shall now construct φ_2 differently.

Let $n > 0$ be an integer, and let $\alpha = 2\pi/n$. We define $\theta_j = (j - \frac{3}{2})\,\alpha$, $j = 1, 2,..., n$, and

$$E_1^{(j)} = \{e^{i\theta} : \theta_j \leqslant \theta \leqslant \theta_{j+1}\}, \qquad E_2^{(j)} = T - E_1^{(j)}. \tag{2.45}$$

In the notation of Theorem 2.4, we take $\lambda_1 = 1$, $\lambda_2 = -1$. Then in the notation of Fig. 1 and (2.3), φ_1/φ_2 takes the special form

$$\varphi^{(j)}(z) = \frac{i(1 + i\sqrt{k}\,\mathrm{sn}[K\{1 + (\alpha - 2\beta_j/\pi) + 2i/\pi \ln b_{j+1}/b_j\}; k])}{1 - i\sqrt{k}\,\mathrm{sn}[K\{1 + (\alpha - 2\beta_j/\pi) + 2i/\pi \ln b_{j+1}/b_j\}; k]}, \tag{2.46}$$

where $E_1 = E_1^{(j)}$, $E_2 = T - E_1$, and where

$$b_j = |\, e^{i\theta_j} - z \,|, \qquad \beta_j = \arg\left(\frac{e^{i\theta_{j+1}} - z}{e^{i\theta_j} - z}\right). \tag{2.47}$$

The poles $z = p_m^{(j)}$ of $\varphi_j(z)$ are given by the solutions of

$$K\left\{1 + \frac{\alpha - 2\beta_j}{\pi} + \frac{2i}{\pi}\ln\frac{b_{j+1}}{b_j}\right\} = 4sK - (2m + \tfrac{1}{2})\,iK',$$
$$m, s = 0, \pm 1, \pm 2,..., \tag{2.48}$$

and are explicitly expressed by

$$p_m^{(j)} = e^{i(j-\frac{1}{2})\alpha}\,\frac{(1 - iq^{\frac{1}{4}+m}e^{-i\alpha/2})}{(1 - iq^{\frac{1}{4}+m}e^{i\alpha/2})},$$
$$m = 0, \pm 1, \pm 2,..., \quad j = 1, 2,..., n, \tag{2.49}$$

where

$$q = e^{-\pi K'/K}. \tag{2.50}$$

It is thus clear that all the poles are simple.

We next construct a function $\Phi_{2,n}(z)$ which is analytic in U and which has a simple zero at each simple pole of $\varphi^{(j)}(z)$. Such a $\Phi_{2,n}(z)$ is given explicitly by

$$\Phi_{2,n}(z) = \prod_{m=-\infty}^{\infty}\prod_{j=1}^{n}\frac{z - p_m^{(j)}}{1 - z\bar{p}_m^{(j)}}. \tag{2.51}$$

It is easily seen that the product in (2.51) converges for every $z \in \bar{U}$. In fact, we may write

$$\Phi_{2,n}(z) = \prod_{m=-\infty}^{\infty} \frac{z^n - (-1)^n \gamma_m^{(n)}}{1 - (-1)^n \bar{\gamma}_m^{(n)} z^n}, \tag{2.52}$$

where

$$\gamma_m^{(n)} = \prod_{j=1}^{n} p_m^{(j)} = (-1)^n \frac{(1 - iq^{\frac{1}{2}+m} e^{-i\alpha/2})^n}{(1 - iq^{\frac{1}{2}+m} e^{i\alpha/2})^n}. \tag{2.53}$$

It is readily seen that

$$\sum_{m=-\infty}^{\infty} [1 - |\gamma_m^{(n)}|] \tag{2.54}$$

converges absolutely, so that (2.52) converges. The property $|\Phi_{2,n}(e^{i\theta})| = 1$ is also a consequence of the definition: each ratio on the right in (2.51) has modulus 1 on T.

COROLLARY 2.9. *Let f be continuous on T. Given $\delta > 0$, there exists a linear combination $G^{(n)}(z)$ of ratios of inner functions such that*

$$|f(e^{i\theta}) - G^{(n)}(e^{i\theta})| < \delta \tag{2.55}$$

for all $e^{i\theta}$ on T.

Proof. Let us subdivide T into n disjoint subsets $E_s^{(j)}$ ($j = 1, 2,..., n$; $s = 1, 2$), given by (2.45), such that

$$\max_{u,v \in E_1^{(j)}} |f(u) - f(v)| < \tfrac{1}{2}\delta, \qquad j = 1, 2,..., n. \tag{2.56}$$

Set $f_j = f(e^{i(j-1)\alpha})$. Since

$$\varphi^{(j)}(z) = \frac{\varphi^{(j)}(z)\varphi_2^{(j)}(z)}{\varphi_2^{(j)}(z)}, \tag{2.57}$$

where $\varphi^{(j)}(z)$ and $\varphi_2^{(j)}(z)$ are defined by (2.46) and (2.51), $\varphi^{(j)}(z)$ is clearly a ratio of inner functions. If, in the notation of Theorem 2.4, we take $\eta = \delta/(Mn)$, $E_1 = E_1^{(j)}$, $E_2 = E_2^{(j)}$, then $|\varphi^{(j)}(e^{i\theta}) - 1| < \eta$ on E_1, $|\varphi^{(j)}(e^{i\theta}) + 1| < \eta$ on E_2. The function $G^{(n)}(z)$, defined by

$$G^{(n)}(z) = \frac{1}{2} \sum_{j=1}^{n} f_j[\varphi^{(j)}(z) + 1]$$
$$= \frac{\frac{1}{2} \sum_{j=1}^{n} f_j[\varphi^{(j)}(z) + 1]\, \Phi_{2,n}(z)}{\Phi_{n,2}(z)}, \tag{2.58}$$

where $\Phi_{2,n}(z)$ is defined by (2.52), is clearly a linear combination of ratios of inner functions which satisfies for $e^{i\theta} \in E_1^{(j)}$:

$$|f(e^{i\theta}) - \lim_{r\to 1^-} G^{(n)}(re^{i\theta})| \leqslant |f(e^{i\theta}) - f_j| + \frac{1}{2}\sum |f_j| \eta$$
$$< \frac{\delta}{2} + \frac{1}{2} Mn\eta < \delta. \tag{2.59}$$

A different constructive proof is possible by proceeding along the lines of the proof of Theorem 2.8. We omit this, however.

In the following theorem, an inner function is called *singular* if it has no zero in U.

THEOREM 2.10 (Douglas–Rudin). *Let $f \in L^\infty(T)$ and let $\epsilon_1 > 0$ be given. Then there exists a singular inner function φ_2 and a finite linear combination ψ_2 of inner functions such that*

$$\| f - (\psi_2/\varphi_2)\|_\infty < \epsilon_1 . \tag{2.60}$$

Proof. By Theorem 2.7, given $\epsilon > 0$, there exist an inner function φ and a linear combination ψ of inner functions such that

$$\| f - \psi/\varphi \|_\infty < \epsilon. \tag{2.61}$$

In fact, with $G^{(n,N)}(z)(G^{(n)}(z))$ defined in (2.37) ((2.58)), we may take $\varphi = \prod_{r,s} \psi_{r,s}(\varphi = \prod_r \psi_r)$ and $\psi = G^{(n,N)}/\varphi$ $(\psi = G^{(n)}/\varphi)$.

The function φ has zeros in U which are removed by the following device used in [3]. Define $u(w)$ by

$$u(w) = \exp\left[c\,\frac{w+1}{w-1}\right], \qquad \ln \epsilon_2^{-1/3} < c < \infty, \tag{2.62}$$

where $\epsilon_2 = \epsilon_1/(2M + 2\epsilon)$, and set

$$u_1(w) = \frac{u(w) - e^{-3c}}{w[1 - e^{-3c}u(w)]} . \tag{2.63}$$

Then $u_1(w)$ is an inner function, and clearly

$$| u(w) - wu_1(w)| < \epsilon_2 , \qquad w \in U. \tag{2.64}$$

We now set $w = \varphi(z)$ in (2.64) and define the compositions

$$\begin{aligned} h_1 &= u_1 \circ \varphi, \\ \psi_2 &= u_2 \circ \varphi. \end{aligned} \tag{2.65}$$

Then h_1 and ψ_2 are inner, ψ_2 has no zero in U and

$$| \psi_2(z) - \varphi(z)\, h_1(z)| < \epsilon_2 , \qquad z \in U. \tag{2.66}$$

Taking radial limits and dividing by $\psi_2\varphi$, we get

$$\left\| \frac{1}{\varphi} - \frac{h_1}{\psi_2} \right\|_\infty < \epsilon_2 . \tag{2.67}$$

To complete the proof, we take $\delta = \epsilon_1/2$ in Theorem 2.7 (or Corollary 2.9) and replace $1/\varphi$ by $h_1\psi_2$. Then we define $\varphi_2(z)$ by

$$\begin{aligned}\varphi_2(z) &= G^{(n,N)}(z)\varphi(z)\,h_1(z)\\ &= \frac{1}{2}\sum_{r,s} f_{r,s}\Big\{\prod_{\substack{r'\neq r\\ s'\neq s}} \psi_{r's'}[\varphi_{r,s} + \psi_{r,s}]\Big\}\\ &\quad\cdot \frac{\exp\left\{c\,\dfrac{(\prod_{r',s'}\psi_{r',s'})+1}{(\prod_{r',s'}\psi_{r',s'})-1}\right\} - e^{-3c}}{\prod_{r',s'}\psi_{r',s'}\left[1 - e^{-3c}\exp\left\{c\,\dfrac{(\prod_{r',s'}\psi_{r',s'})+1}{(\prod_{r',s'}\psi_{r',s'})-1}\right\}\right]} .\end{aligned} \tag{2.68}$$

Analogous results hold, corresponding to Theorem 2.8 and Corollary 2.9.

It seems natural to investigate what happens to some of the above approximate expressions as the error approaches zero. It would be interesting, for example, to study what happens to the functions $\varphi_j(z)$ in (2.22) and (2.23), and to $\Phi_{2,n}(z)$ in (2.51), as $k \to 1$. This does not appear to be trivial.

It is known, for example [4], that there exist functions $\varphi_E(z)$ and $\psi_E(z)$, both analytic in U, such that

$$\lim_{r\to 1^-} \frac{\varphi_E(re^{i\theta})}{\psi_E(re^{i\theta})} = \begin{cases} 1 & \text{if } e^{i\theta} \in E,\\ 0 & \text{if } e^{i\theta} \in T - E,\end{cases} \tag{2.69}$$

a.e. on T. However, it is clear that there do not exist inner functions φ_1, φ_2, ψ_1, ψ_2 such that

$$\lim_{r\to 1^-} \frac{1}{2}\left\{\frac{\varphi_1(re^{i\theta})}{\psi_1(re^{i\theta})} + \frac{\varphi_2(re^{i\theta})}{\psi_2(re^{i\theta})}\right\} = \begin{cases} 1 & \text{if } e^{i\theta} \in E,\\ 0 & \text{if } e^{i\theta} \in T - E,\end{cases} \tag{2.70}$$

a.e. on T, unless either E or $T - E$ has measure zero.

3. Remarks on the Approximate Solution of Wiener–Hopf Equations

Let R denote the real line, and consider the equation

$$f(x) = \int_0^\infty k(x-t)\,f(t)\,dt + g(x), \quad x > 0, \tag{3.1}$$

where $k, g \in L^1(R)$. If, for every such given g, (3.1) has a solution f, then this solution may be found by the classical Wiener–Hopf technique [1]. The chief difficulty of carrying this out in practice is finding functions K_+ and K_- such that the equation

$$(1 - K)^{-1} = (1 + K_+)(1 + K_-) \tag{3.2}$$

holds everywhere on R, where

$$\begin{aligned} K(x) &= \int_R e^{ixt} k(t)\, dt, \\ K_+(x) &= \int_0^\infty e^{ixt} k_1(t)\, dt, \\ K_-(x) &= \int_{-\infty}^0 e^{ixt} k_2(t)\, dt \end{aligned} \tag{3.3}$$

and where $k_1, k_2 \in L^1(R)$. The function $K_+(x + iy)(K_-(x + iy))$ is analytic and bounded in $\{x + iy \mid y \geqslant 0\}(\{x + iy \mid y \leqslant 0\})$. Since direct approximate methods for solving (3.1) are sorely lacking [2] we are tempted to apply the technique developed in Section 2.

The transformation

$$z = i\frac{1 - w}{1 + w} \qquad (w = u + iv,\ z = x + iy) \tag{3.4}$$

maps the upper half of the z-plane conformally onto $| w | < 1$, while the real line R $(-\infty < x < \infty)$ is mapped in a (1, 1) manner onto $| w | = 1$. The function κ, defined by $\kappa(w) = [1 - K(z(w))]^{-1}$, is thus in $L^\infty(T)$, and we can apply the analysis of Section 2 to obtain an approximate representation of κ as a ratio of functions analytic in $| w | < 1$ and a fortiori to obtain an approximate factorization of the form (3.2).

This outlined procedure has indeed been carried out yielding an approximate solution of the equation

$$f(t) = \frac{\mu}{2\pi} \int_0^\infty \frac{f(s)}{\cosh((t - s)/2)}\, ds$$

whose (exact) solution is known [1]; this approximate solution turned out to be a very good approximation. However, since we have not been able to establish that the approximate representations obtained in Section 2 converge, as $\epsilon \to 0$, we have not been able, in general, to establish the convergence of the approximate solution of (3.1) obtained by this technique. We have thus chosen not to include here the details of this approximation method.

In another paper [5] which was motivated by the present one, the author derives a direct method (i.e., without the use of inner functions) of obtaining an approximate factorization of the type (3.2), which converges to the unique factorization, whenever a unique factorization exists. In [5] it is assumed that $k, g \in L^1(R) \cap L^2(R)$, and it is shown that the approximate solution of the equation (3.1) obtained via the approximate factorization of the form (3.2) converges to the exact solution.

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